MATHEMATICS

# FINITE GROUPS WITH AT MOST $p+1$ SYLOW $p$-SUBGROUPS

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All groups considered will be finite. It is well-known that a group with exactly one Sylow $p$-subgroup for every prime divisor $p$ of its order is the direct product of its Sylow subgroups, i.e. is nilpotent. The aim of this note is to prove a structure theorem for those groups which have exactly $p+1$ Sylow $p$-subgroups for every prime divisor $p$ of their orders; let us call such groups *M-groups.*

Examples of $M$-groups are $\mathfrak{S}_4$ and $\mathfrak{A}_4 \times \mathfrak{S}_3$; here $\mathfrak{S}_n$ and $\mathfrak{A}_n$ are the symmetric and alternating groups of $n$ objects, respectively. If $p$ is a Mersenne prime, i.e. a prime of the form $2^k-1$, then an $M$-group can be constructed as follows. Let $V_{p+1}$ be an elementary abelian group with $p+1$ elements (every element has order $\leqslant 2$). Then the automorphism group of $V_{p+1}$ is isomorphic to the multiplicative group of all non-singular $k \times k$-matrices, with coefficients taken from the field of order 2. Since the order of this automorphism group is divisible by $p$, $V_{p+1}$ has an automorphism of order $p$. So we can form a splitting extension $E_p$ of $V_{p+1}$ by means of such an automorphism. It is clear that $E_p$ has exactly one Sylow 2-subgroup and $p+1$ Sylow $p$-subgroups. For instance, $E_3 \cong \mathfrak{A}_4$. Now if $A$ is a direct product of some distinct groups $E_p$, for $p>3$, then $\mathfrak{S}_4 \times A$ and $\mathfrak{A}_4 \times \mathfrak{S}_3 \times A$ are examples of $M$-groups. Our theorem is:

Theorem. *Let $G$ be a finite $M$-group with more than one element. Then every odd prime divisor of $|G|$ is a Mersenne prime, and there exists a nilpotent normal subgroup $N$ of $G$ and a group $A$ as described above such that either $G/N \cong \mathfrak{S}_4 \times A$ or $G/N \cong \mathfrak{A}_4 \times \mathfrak{S}_3 \times A$.*

The proof will be split up in a number of lemmas.

Lemma 1. Let $p$ be a prime, and $G$ a group with exactly $pv+1$ Sylow $p$-subgroups, where $0 \leqslant v < p$. Let $P$ be a Sylow $p$-group of $G$, and $N_p$ the maximal normal $p$-subgroup of $G$ (i.e. the intersection of all Sylow $p$-subgroups of $G$). Then $P/N_p$ is elementary abelian of order $\leqslant p^v$.

Proof. Let $\mathfrak{N}(P)$ be the normalizer of $P$, and $P^a$ a Sylow $p$-subgroup of $G$ distinct from $P$. Then $\mathfrak{N}(P) \cap P^a = P \cap P^a < P$. Now in $P^a$, distinct (right) cosets to $P \cap P^a$ produce distinct conjugates of $P$. Hence $p \leqslant [P^a : P \cap P^a] \leqslant vp+1$. Since $[P^a : P \cap P^a]$ is a $p$-power and $vp+1 < p^2$, it follows: $[P^a : P \cap P^a]=p$. Being a subgroup of index $p$ in the $p$-groups $P$ and $P^a$, $P \cap P^a$ is normal in $P$ and $P^a$. It follows that $\mathfrak{N}(P \cap P^a)$ contains $P$ and $P^a$, so at least $p+1$ Sylow $p$-subgroups of $G$. So if we bring the conjugates $P^a$ and $P^b$ distinct from $P$ in the same class iff $P \cap P^a = P \cap P^b$, every class contains at least $p$ elements. So the number $w$ of these classes is at most $v$. If $P^{a_1}, \ldots, P^{a_w}$ are representatives from these classes, then

$$N_p = \bigcap_{i=1}^{w} P \cap P^{a_i}.$$

It is clear that $[P : N_p] \leqslant p^w \leqslant p^v$; $P/N_p$ is also elementary abelian since $P/P \cap P^{a_i}$ is elementary abelian for each $i$.

Lemma 2. Let $G$ be an $M$-group, and $N_p$ its maximal normal $p$-subgroup ($p$ prime). Then:

(i) $G$ is soluble;
(ii) every odd prime divisor of $|G|$ is a Mersenne prime;
(iii) $G/\prod_{p>3} N_p$ is still an $M$-group.

Proof. Obviously, the group generated by the groups $N_p$ is equal to the direct product $\prod_p N_p$ of them, and is normal in $G$. Moreover, $\prod_p N_p$ is nilpotent, whence soluble. The group $G/\prod_p N_p$ has by lemma 1 all its Sylow subgroups cyclic, even of prime orders. By a theorem of W. Burnside (see e.g. [3, p. 175]), a group with cyclic Sylow subgroups is soluble. Hence $G$ is soluble as well. The group $G$ being soluble, by a theorem of P. Hall [2, p. 318], it has a complete system of mutually permutable Sylow subgroups $S_p$ (where $p$ runs through the set of prime divisors of $|G|$). This means that for different prime divisors $p$, $q$ of $|G|$, $S_pS_q = S_qS_p$. The number of Sylow $p$-subgroups of $S_pS_q$ is a power of $q$ and also either 1 or $p+1$. It follows that for odd $p$, $q$ this number is 1; so for those $p, q$ $S_p$ and $S_q$ normalize each other, and thus also centralize each other. Thus for odd $p$, $\mathfrak{N}(S_p)$ contains all $S_q$, for $q$ odd; it follows that $p+1=[G : \mathfrak{N}(S_p)]$ can only be a power of 2. This shows that all odd prime divisors of $|G|$ are Mersenne primes. Also $\mathfrak{N}(S_2) \geqslant S_q$ $(q>3)$, since $\mathfrak{N}(S_2)$ has index 3 in $G$. So, in particular, every $S_p$ contains $N_q$ $(q>3,\ q \neq p)$ in its centralizer. From this it readily follows that the number of Sylow subgroups of $G/\prod_{p>3} N_p$ is not less than that of $G$, which proves (iii).

Lemma 3. Let $G$ be an $M$-group, $N_2$ the maximal normal 2-subgroup of $G$, $S_p$ a Sylow $p$-subgroup for an odd prime divisor $p$ of $|G|$, and $T_p = N_2 \cap \mathfrak{N}(S_p)$. Then:

(i) $T_p$ is a normal subgroup of any Sylow 2-subgroup of $G$, and $N_2/T_p$ is elementary abelian of order $p+1$;

(ii) $T = \bigcap_{p \geqslant 3} T_p$ is a normal subgroup of $G$, $G/T$ is an $M$-group, and $N_2/T$ is elementary abelian order $\leqslant \prod\{p+1 \mid p \geqslant 3, p \mid |G|\}$.

Proof. By P. Hall [2, p. 321], any given Sylow $p$-subgroup can be included in a complete system of mutually permutable Sylow subgroups (if the whole group is soluble). So let $\{S_q\}_q$ be such a system which includes the Sylow 2-subgroup mentioned in (i). The Sylow $p$-subgroup in this system need not coincide with the given Sylow $p$-subgroup, but we shall assume it does. In fact, $T_p$ will be independent of the particular Sylow $p$-subgroup $S_p$ chosen. This follows from the normality of $T_p$ in $N_2$, and the fact that conjugation of $S_p$ by means of elements of $N_2 \setminus T_p$ yields all Sylow $p$-subgroups of $G$.

Since $[G : \mathfrak{N}(S_p)] = p+1 > 2$ and $[S_2 : \mathfrak{N}(S_p) \cap S_2] = [G : \mathfrak{N}(S_p)]$ because $\mathfrak{N}(S_p)$ contains $S_q$ for $q > 2$, we necessarily have: $T_p \neq N_2$. It follows that the group $N_2 S_p$ has more than one Sylow $p$-subgroup, so exactly $p+1 = [N_2 : T_p]$. Now $T_p = N_2 \cap \mathfrak{N}(S_p)$ being contained in the normal subgroup $N_2$ which has trivial intersection with $S_p$, $T_p$ centralizes $S_p$. So if we conjugate a maximal subgroup $H$ of $N_2$ containing $T_p$ by the elements of $S_p$, the intersection $I$ of the conjugates of $H$ contains $T_p$, and is normal in $N_2$, being an intersection of subgroups of index 2 in $N_2$. In $N_2/I$ nontrivial automorphisms are induced by the $p$-group $S_p$, whence $N_2/I$ has at least $p+1$ elements. This implies $I = T_p$, so $T_p$ is normal in $N_2$. Since $N_2/T_p$ contains an orbit of length $p$, $N_2/T_p$ is elementary abelian. This shows (i).

Since $\mathfrak{N}(S_p) \cap S_2$ must also contain elements outside $N_2$, $T_p$ is a subgroup of index 2 in it, so also normal in it. This implies that $T_p$ is normal in $S_2$. Hence $T = \bigcap_{p \geqslant 3} T_p$ is also normal in $S_2$. But $T$ also centralizes all groups $S_p$ $(p \geqslant 3)$, in fact all Sylow $p$-subgroups of $G$ for $p \geqslant 3$. This implies that $T$ is normal in $G$ and that $G/T$ is again an $M$-group. The last assertion of the lemma is clear.

Lemma 4. Let $G$ be an $M$-group of which all Sylow $p$-subgroups are cyclic of order $p$ for all $p > 3$, $p \mid |G|$. Moreover, let its maximal normal 2-group $N_2$ be elementary abelian of order $\leqslant \prod\{p+1 \mid p \geqslant 3, p \mid |G|\}$. Then there is a normal subgroup $B$ of $G$ such that:

(i) $G \cong B \times \prod\{E_p \mid p > 3, p \mid |G|\}$,

(ii) $B$ is an $M$-group, $|B|$ has only 2 and 3 as prime divisors, and the maximal normal 2-group of $B$ is elementary abelian of order 4.

Proof. Again, let $\{S_p\}_p$ be a complete system of mutually permutable Sylow subgroups of $G$. Let $U$ be the subgroup generated by the groups $S_p$ $(p \geqslant 3)$. Since those groups centralize each other, $U$ is in fact the direct product of them. We again put $T_p = N_2 \cap \mathfrak{N}(S_p)$ $(p \geqslant 3)$; we noticed that $T_p$ consists precisely of those elements of $N_2$ that centralize $S_p$. Since

$S_p$ and $S_q$ centralize each other for any $q \geqslant 3$, it easily follows that $T_p$ is invariant under $U$. Now the representation of $U$ on the linear space $N_2$ over the field of 2 elements is completely reducible since $2 \nmid |U|$ (see e.g. M. HALL [1, p. 253]). Hence $N_2 = T_p \times L_p$, where $L_p$ is invariant under $U$ and $|L_p| = p+1$. Since $S_p$ already acts transitively on the elements $\neq e$ of $L_p$, $L_p$ is an irreducible $U$-module. Hence there are irreducible $U$-modules of order $p+1$ for every $p \geqslant 3$. So $|N_2| \geqslant \prod \{p+1 \mid p \geqslant 3,\ p \mid |G|\}$, and also the equality is true. Moreover, $N_2 = \prod\limits_{p \geqslant 3} L_p$. It is easy to verify that the group $W$ generated by the groups $S_p$, $L_p$ for $p > 3$, $p \mid |G|$ is isomorphic to $\prod \{E_p \mid p > 3,\ p \mid |G|\}$. It is also true that $W$ is centralized by $L_3$ and $S_3$. Further, if $a \in S_2 \setminus N_2$, then for every $p \geqslant 3$ there exists an element $b_p \in N_2$ such that $ab_p$ normalizes $S_p$. Since $S_2$ is normal in $S_2S_p$, every element which normalizes $S_p$, also centralizes it ($p > 3$). So $ab_p$ centralizes $S_p$ ($p > 3$). Since $T_p$ centralizes $S_p$, we may assume that $b_p \in L_p$. This ensues that $b_p$ centralizes $S_q$, for $q \neq p$, $q \geqslant 3$. Finally, $c = a \prod\limits_{p \geqslant 3} b_p$ centralizes $S_p$ for all $p > 3$, and normalizes $S_3$. The same applies to $c^2$, and therefore $c^2$, being an element of $N_2$, can only be $e$. We obtain a decomposition as required by taking for $B$ the group generated by $L_3$, $S_3$ and $c$.

Lemma 5. Let $G$ be an $M$-group, with $N_2$ and $N_3$ as maximal normal 2-subgroup and 3-subgroup respectively, and $|N_2| = 4$. Assume $G/N_2N_3 \cong \mathfrak{S}_3$. Then $G/N_3 \cong \mathfrak{S}_4$.

Proof. Let $S_2$ and $S_3$ be a Sylow 2-subgroup and 3-subgroup, respectively. They are necessarily permutable since $S_2S_3 = G$, as follows from the hypotheses. From lemma 3, we know $N_2$ does not normalize $S_3$. This implies that $G/N_3$ still has 4 Sylow 3-subgroups. Since even $G/N_2N_3$ has 3 Sylow 2-subgroups, $G/N_3$ certainly has. So $G/N_3$ is an $M$-group. For convenience, we assume $N_3 = \{e\}$. Now $|G| = 24$.

An element $d \in S_2 \setminus N_2$ cannot centralize $S_3$, because otherwise $S_3$ would normalize the group generated by $N_2$ and $d$, i.e. $S_2$, and $S_2$ would be normal in $G$. Then $d$ cannot centralize $N_2$ either, thus $S_2$ is non-abelian. Since $S_2$ contains already 3 elements of order 2 in the elementary abelian $N_2$, $S_2$ is the dihedral group of 8 elements. Because $N_2S_3$ is of index 2 in $G$, it is normal in $G$. Moreover, $S_2$ contains elements of order 2 outside $N_2$. This shows that $G$ is a non-trivial splitting extension of $N_2S_3$ by an automorphism of order 2; since $N_2S_3 \cong \mathfrak{A}_4$, it follows that $G \cong \mathfrak{S}_4$.

Lemma 6. Let $G$ be an $M$-group, with $N_2$ and $N_3$ as maximal normal 2-subgroup and 3-subgroup, respectively, and $|N_2| = 4$. Assume $G/N_2N_3 \cong \mathfrak{C}_6$ (cyclic group of order 6). Then $G/N_3 \cap \mathfrak{N}(S_2) \cong \mathfrak{A}_4 \times \mathfrak{S}_3$.

Proof. Let $S_2$ and $S_3$ be a Sylow 2-subgroup and 3-subgroup, respectively. As in the proof of lemma 5, it is clear that $G/N_3 \cap \mathfrak{N}(S_2)$ has

4 Sylow 3-subgroups. It follows from $G/N_2N_3 \cong \mathfrak{C}_6$, that $S_2N_3$ is normal in $G$. Then $S_2$ cannot be normal in $S_2N_3$ since otherwise it would be characteristic in it, so certainly normal in $G$. So $[N_3 : \mathfrak{N}(S_2) \cap N_3]=3$. Since $\mathfrak{N}(S_2) \cap N_3$ even centralizes $S_2$, it follows that $G/N_3 \cap \mathfrak{N}(S_2)$ still has 3 Sylow 2-subgroups. We assume that $N_3 \cap \mathfrak{N}(S_2)=\{e\}$. Then $|G|=72$. Now $S_3=N_3 \times (S_3 \cap \mathfrak{N}(S_2))$, and $\mathfrak{N}(S_3)=D \times (S_3 \cap \mathfrak{N}(S_2))$, where $D$ is the subgroup generated by $N_3$ and an element $f$ of order 2 of $S_2 \setminus N_2$ that normalizes $S_3$ (it necessarily centralizes $\mathfrak{N}(S_2)$); $D \cong \mathfrak{S}_3$.

Further, $N_2$ centralizes $N_3$, and also $f$; hence $N_2$ is central in $S_2$, and $S_2$ is abelian. But $N_2(S_3 \cap \mathfrak{N}(S_2)) \cong \mathfrak{A}_4$ and $G=D \times N_2(S_3 \cap \mathfrak{N}(S_2))$; this proves the lemma.

The theorem follows by the consecutive application of the lemmas; the normal subgroup $N$ is a subgroup of

$$\prod \{N_p| \ p| \ |G|\}.$$

Remark. The discussion, in terms of the theorem proved, of the groups which have at most $p+1$ Sylow $p$-subgroups for all prime divisor of their orders, presents no difficulties. Again such groups are soluble. Normal Sylow $p$-subgroups for $p>3$ are always direct factors. If we restrict ourselves to such groups, which are non-$M$-groups and which have no primary direct factors, then either (i) the Sylow 2-subgroup is normal and a Sylow 3-subgroup is not or (ii) a Sylow 2-subgroup is not normal and the Sylow 3-subgroup is. In case (i) we obtain: $G/N \cong \mathfrak{A}_4 \times A$, and in case (ii): $G/N \cong \mathfrak{S}_3 \times A$, where $N$ and $A$ are as in the main theorem.

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